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An Equivalent Infinitesimal Formula in Fractional Calculus and Its Applications

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Abstract: **Based on Jumarie's modified Riemann-Liouville (R-L) fractional derivative, this paper studies an equivalent infinitesimal formula in fractional calculus. Fractional L'Hospital's rule and a new multiplication of fractional analytic functions play important roles in this article. On the other hand, we give some examples to illustrate the applications of this formula. In fact, our results are generalizations of classical calculus results.**

Keyword: **Jumarie's modified R-L fractional derivative, equivalent infinitesimal formula, fractional L'Hospital's rule, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus is the theory of derivative and integral of non-integer order, which can be traced back to Leibniz, Liouville, Grunwald, Letnikov and Riemann. Fractional calculus has been attracting the attention of scientists and engineers from long time ago, and has been widely used in physics, engineering, biology, economics and other fields [1-15]. The definition of fractional derivative is not unique. The commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [16-19]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with ordinary calculus.

In this paper, based on Jumarie's modified R-L fractional derivative, we study an equivalent infinitesimal formula in fractional calculus. Fractional L'Hospital's rule and a new multiplication of fractional analytic functions play important roles in this paper. Moreover, two examples are provided to illustrate the applications of this formula. In fact, our results are generalizations of traditional calculus results.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([20])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,. \tag{1}$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([21])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([22])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x - x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([23])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} . \tag{5}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) (x - x_0)^{n\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha \right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha \right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) \left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha \right)^{\otimes_\alpha n} . \end{aligned} \tag{7}$$

Definition 2.5 ([24])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha \right)^{\otimes_\alpha n} , \tag{8}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha \right)^{\otimes_\alpha n} . \tag{9}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{a_n}{n!} (g_\alpha(x^\alpha))^{\otimes_\alpha n}, \tag{10}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{b_n}{n!} (f_\alpha(x^\alpha))^{\otimes_\alpha n}. \tag{11}$$

Definition 2.6 ([25])**:** Let $0 < \alpha \leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \tag{12}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([26])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $(f_\alpha(x^\alpha))^{\otimes_\alpha n} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $(f_\alpha(x^\alpha))^{\otimes_\alpha (-1)}$.

Definition 2.8 ([27])**:** If $0 < \alpha \leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha n}. \tag{13}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha 2n}, \tag{14}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{15}$$

Definition 2.9 ([28])**:** Let $0 < \alpha \leq 1$. If $u_\alpha(x^\alpha)$, $w_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then the α-fractional power exponential function $u_\alpha(x^\alpha)^{\otimes_\alpha w_\alpha(x^\alpha)}$ is defined by

$$u_\alpha(x^\alpha)^{\otimes_\alpha w_\alpha(x^\alpha)} = E_\alpha\left(w_\alpha(x^\alpha) \otimes_\alpha Ln_\alpha(u_\alpha(x^\alpha))\right). \tag{16}$$

Theorem 2.10 (fractional L'Hospital's rule) ([29])**:** *Assume that* $0 < \alpha \leq 1$, c *is a real number, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ $[g_\alpha(x^\alpha)]^{\otimes_\alpha (-1)}$ *are* α*-fractional analytic functions at* $x = c$. *If* $\lim_{x\to c} f_\alpha(x^\alpha) = \lim_{x\to c} g_\alpha(x^\alpha) = 0$, *or* $\lim_{x\to c} f_\alpha(x^\alpha) = \pm\infty$, *and* $\lim_{x\to c} g_\alpha(x^\alpha) = \pm\infty$. *Suppose that* $\lim_{x\to c} f_\alpha(x^\alpha) \otimes_\alpha [g_\alpha(x^\alpha)]^{\otimes -1}$ *and* $\lim_{x\to c} \left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes_\alpha \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes_\alpha (-1)}$ *exist,* $\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c) \neq 0$. *Then*

$$\lim_{x\to c} f_\alpha(x^\alpha) \otimes_\alpha [g_\alpha(x^\alpha)]^{\otimes_\alpha (-1)} = \lim_{x\to c} \left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes_\alpha \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes_\alpha (-1)}. \tag{17}$$

Definition 2.11: Let $0 < \alpha \leq 1$. Suppose that c is a real number and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic functions such that $\lim_{x\to c} f_\alpha(x^\alpha) = \lim_{x\to c} g_\alpha(x^\alpha) = 0$ and $\lim_{x\to c} f_\alpha(x^\alpha) \otimes_\alpha [g_\alpha(x^\alpha)]^{\otimes_\alpha (-1)} = 1$, then we say that $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are equivalent infinitesimal when $x \to c$, and denoted by $f_\alpha(x^\alpha) \sim g_\alpha(x^\alpha)$.

III. RESULTS AND EXAMPLES

In this section, we study an equivalent infinitesimal formula in fractional calculus and two examples are provided to illustrate its applications. At first, we need two lemmas.

Lemma 3.1: *Suppose that* $0 < \alpha \leq 1$, c *is a real number and* $g_\alpha(x^\alpha)$ *is a* α*-fractional analytic function such that* $\lim_{x\to c} g_\alpha(x^\alpha) = 0$. *Then*

$$Ln_\alpha(1 + g_\alpha(x^\alpha)) \sim g_\alpha(x^\alpha), \tag{18}$$

when $x \to c$.

Proof Since $\lim_{x\to c} g_\alpha(x^\alpha) = 0$ and by fractional L'Hospital's rule,

$$\lim_{x\to c} Ln_\alpha(1 + g_\alpha(x^\alpha)) \otimes_\alpha [g_\alpha(x^\alpha)]^{\otimes_\alpha (-1)}$$

$$= \lim_{x\to c} \left({}_cD_x^\alpha\right)[Ln_\alpha(1 + g_\alpha(x^\alpha))] \otimes_\alpha \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes_\alpha (-1)}$$

$$= \lim_{x\to c} [[1 + g_\alpha(x^\alpha)]]^{\otimes_\alpha (-1)} \otimes_\alpha \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right] \otimes_\alpha \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes_\alpha (-1)}$$

$$= \lim_{x\to c} [1 + g_\alpha(x^\alpha)]^{\otimes_\alpha (-1)}$$

$$= \left[1 + \lim_{x\to c} g_\alpha(x^\alpha)\right]^{\otimes_\alpha (-1)}$$

$$= 1. \tag{19}$$

It follows from Definition 2.10 that $Ln_\alpha\big(1+g_\alpha(x^\alpha)\big) \sim g_\alpha(x^\alpha)$, when $x \to c$. Q.e.d.

Lemma 3.2: *If* $0<\alpha\le 1$. *Assume that* c *is a real number and* $h_\alpha(x^\alpha)$ *is a* α*-fractional analytic function such that* $\lim_{x\to c} h_\alpha(x^\alpha)=0$. *Then*

$$E_\alpha\big(h_\alpha(x^\alpha)\big)-1 \sim h_\alpha(x^\alpha), \tag{20}$$

when $x \to c$.

Proof By $\lim_{x\to c} h(x^\alpha)=0$ and fractional L'Hospital's rule, we have

$$\lim_{x\to c}\big[E_\alpha\big(h_\alpha(x^\alpha)\big)-1\big]\otimes_\alpha [h_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}$$

$$=\lim_{x\to c}\big({}_cD_x^\alpha\big)\big[E_\alpha\big(h_\alpha(x^\alpha)\big)-1\big]\otimes_\alpha \Big[\big({}_cD_x^\alpha\big)[h_\alpha(x^\alpha)]\Big]^{\otimes_\alpha(-1)}$$

$$=\lim_{x\to c} E_\alpha\big(h_\alpha(x^\alpha)\big)\otimes_\alpha \Big[\big({}_cD_x^\alpha\big)[h_\alpha(x^\alpha)]\Big]\otimes_\alpha \Big[\big({}_cD_x^\alpha\big)[h_\alpha(x^\alpha)]\Big]^{\otimes_\alpha(-1)}$$

$$=\lim_{x\to c} E_\alpha\big(h_\alpha(x^\alpha)\big)$$

$$=E_\alpha\Big(\lim_{x\to c} h_\alpha(x^\alpha)\Big)$$

$$=E_\alpha(0)$$

$$=1\,. \tag{21}$$

It follows that $E_\alpha\big(h_\alpha(x^\alpha)\big)-1 \sim h_\alpha(x^\alpha)$ when $x \to c$. Q.e.d.

Theorem 3.3: *Let* $0<\alpha\le 1$. *If* c *is a real number and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions such that* $\lim_{x\to c} g_\alpha(x^\alpha)=0$, *and* $\lim_{x\to c} f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)=0$. *Then*

$$\big(1+g_\alpha(x^\alpha)\big)^{\otimes_\alpha f_\alpha(x^\alpha)}-1 \sim f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha), \tag{22}$$

when $x \to c$.

Proof Since $\big(1+g_\alpha(x^\alpha)\big)^{\otimes_\alpha f_\alpha(x^\alpha)} = E_\alpha\Big(f_\alpha(x^\alpha)\otimes_\alpha Ln_\alpha\big(1+g_\alpha(x^\alpha)\big)\Big)$, it follows that when $x \to c$,

$$\big(1+g_\alpha(x^\alpha)\big)^{\otimes_\alpha f_\alpha(x^\alpha)}-1$$

$$=E_\alpha\Big(f_\alpha(x^\alpha)\otimes_\alpha Ln_\alpha\big(1+g_\alpha(x^\alpha)\big)\Big)-1$$

$$\sim E_\alpha\Big(f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\Big)-1 \qquad \text{(by Lemma 3.1)}$$

$$\sim f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\,. \qquad \text{(by Lemma 3.2)}$$

Q.e.d.

Example 3.4: Let $0<\alpha\le 1$. Find $\lim_{x\to 0}\left[\left[1-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}\right]^{\otimes_\alpha sin_\alpha(x^\alpha)}-1\right]\otimes_\alpha\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-3)}$.

Solution Since

$$\lim_{x\to 0}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}=0, \text{ and } \lim_{x\to 0}-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}\otimes_\alpha sin_\alpha(x^\alpha)=0.$$

It follows from Theorem 3.3 that

$$\left[1-\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}\right]^{\otimes_\alpha sin_\alpha(x^\alpha)}-1 \sim -\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}\otimes_\alpha sin_\alpha(x^\alpha)\,. \tag{23}$$

Therefore,

$$\lim_{x\to 0}\left[\left[1-\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}\right]^{\otimes_\alpha sin_\alpha(x^\alpha)}-1\right]\otimes_\alpha\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha(-3)}$$

$$=\lim_{x\to 0}\left[-\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}\otimes_\alpha sin_\alpha(x^\alpha)\right]\otimes_\alpha\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha(-3)}$$

$$=\lim_{x\to 0}[-sin_\alpha(x^\alpha)]\otimes_\alpha\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha(-1)}$$

$$=\lim_{x\to 0}[-cos_\alpha(x^\alpha)]\quad\text{(by fractional L'Hospital's rule)}$$

$$=-1\,. \tag{24}$$

Example 3.5: If $0<\alpha\leq 1$. Find $\lim_{x\to 0}\left[[cos_\alpha(x^\alpha)]^{\otimes_\alpha \frac{1}{\Gamma(\alpha+1)}x^\alpha}-1\right]\otimes_\alpha[sin_\alpha(x^\alpha)]^{\otimes_\alpha(-2)}$.

Solution Since

$$[cos_\alpha(x^\alpha)]^{\otimes_\alpha \frac{1}{\Gamma(\alpha+1)}x^\alpha}=[1+[cos_\alpha(x^\alpha)-1]]^{\otimes_\alpha \frac{1}{\Gamma(\alpha+1)}x^\alpha}\,,$$

and

$$\lim_{x\to 0}[cos_\alpha(x^\alpha)-1]=0,\quad \lim_{x\to 0}[cos_\alpha(x^\alpha)-1]\otimes_\alpha\frac{1}{\Gamma(\alpha+1)}x^\alpha=0.$$

It follows from Theorem 3.3 that

$$[cos_\alpha(x^\alpha)]^{\otimes_\alpha \frac{1}{\Gamma(\alpha+1)}x^\alpha}-1\sim[cos_\alpha(x^\alpha)-1]\otimes_\alpha\frac{1}{\Gamma(\alpha+1)}x^\alpha\,. \tag{25}$$

Thus,

$$\lim_{x\to 0}\left[[cos_\alpha(x^\alpha)]^{\otimes_\alpha \frac{1}{\Gamma(\alpha+1)}x^\alpha}-1\right]\otimes_\alpha[sin_\alpha(x^\alpha)]^{\otimes_\alpha(-2)}$$

$$=\lim_{x\to 0}\left[[cos_\alpha(x^\alpha)-1]\otimes_\alpha\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\otimes_\alpha[sin_\alpha(x^\alpha)]^{\otimes_\alpha(-2)}$$

$$=\lim_{x\to 0}\left[[cos_\alpha(x^\alpha)-1]\otimes_\alpha\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\otimes_\alpha\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha(-2)}$$

$$=\lim_{x\to 0}[cos_\alpha(x^\alpha)-1]\otimes_\alpha\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha(-1)}$$

$$=\lim_{x\to 0}[-sin_\alpha(x^\alpha)]\quad\text{(by fractional L'Hospital's rule)}$$

$$=0\,. \tag{26}$$

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional derivative, an equivalent infinitesimal formula in fractional calculus is studied. Fractional L'Hospital's rule and a new multiplication of fractional analytic functions play important roles in this article. In addition, we give two examples to illustrate the applications of this formula. In fact, our results are generalizations of ordinary calculus results. In the future, we will continue to study the problems in applied mathematics and fractional differential equations.

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